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Escape to infinity in a Newtonian potential

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Abstract. Escape to infinity in the presence of a Newtonian potential is examined in the classical and relativistic cases.

1. Introduction

In recent papers [1, 2] the blow-up of $\mathbb{R}^n$ vector fields (VF) has been studied by means of local series around movable singularities (the Painlevé analysis) [3, 4]. In this paper we study a related problem: the escape to infinity (in the configuration space of the coordinates x, y, z) of a particle ruled by Newtonian equations of the following type:

$$\begin{aligned} \ddot{\boldsymbol{x}} &= -\nabla V(\boldsymbol{x}) \\ \boldsymbol{x} &= (x, y, z). \end{aligned} \tag{1}$$

In this paper $V(\boldsymbol{x})$ denotes a finite or infinite superposition of terms of the type $\|\boldsymbol{x}-\boldsymbol{x}_i\|^{-1}$, and these kinds of potentials shall be called Newtonian potentials. We shall also assume that all masses m_i at $\boldsymbol{x}_i$ are stationary in their reference frame.

The time taken by the particle in reaching $\|\boldsymbol{x}\| = +\infty$ can be finite or infinite. However, we are not interested in the boundedness or otherwise of this time.

Our problem is this: assume that $(\boldsymbol{x}_0, \dot{\boldsymbol{x}}_0)$ are the initial conditions of equations (1) and that $\boldsymbol{x}(t, \boldsymbol{x}_0, \dot{\boldsymbol{x}}_0)$ is the corresponding solution. Assume that $\boldsymbol{x}_0$ is fixed but that $\dot{\boldsymbol{x}}_0$ can be changed. Is it possible to reach $\|\boldsymbol{x}\| = +\infty$ by choosing $\dot{\boldsymbol{x}}_0$ appropriately?

Note that when $V(\boldsymbol{x})$ is a central potential our question is rather trivial as equations (1) are integrable [5]. However, when several Newtonian attracting masses are present equations (1) are, in general, non-integrable and the escape to infinity must be analysed in other ways.

Escape to infinity under the presence of a finite number of particles or material bodies is considered in section 2, while section 3 deals with escape in the presence of an infinite number of particles. In section 4 we consider relativistic escape, governed by equations of motion of the form

$$\begin{aligned} m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \dot{\boldsymbol{x}}) &= -\nabla V(\boldsymbol{x}) \\ \gamma &= (1 - \dot{\boldsymbol{x}}^2)^{-1/2} \\ m_0 &= \text{proper mass}. \end{aligned} \tag{2}$$

Finally, in section 5, we end with some open problems.

Escape in the presence of a Newtonian potential has been considered in [6, 7] when the potential is non-Newtonian. A careful mathematical study of Newtonian potentials can be found in [8].

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2. Escape to infinity in the presence of a finite number of masses

Case 2.1

We consider first the case of a finite number N of attracting masses m_i ($i = 1, \ldots, N$) situated at the points $\boldsymbol{x}_i$.

The force exerted by them on a mass $M = 1$ at $\boldsymbol{x} = (x, y, z)$ is

$$\boldsymbol{F}(x, y, z) = -\sum_{i=1}^{N} G \frac{m_i(\boldsymbol{x} - \boldsymbol{x}_i)}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^3} \tag{3}$$

where G denotes the gravitational constant. We assume that $\boldsymbol{x}_i \neq \boldsymbol{0}$, $i = 1, \ldots, N$.

The differential equations of motion of the mass $M = 1$ are

$$\ddot{\boldsymbol{x}} = \boldsymbol{F}(x, y, z) \tag{4}$$

and since

$$\begin{aligned} \rho \cdot \rho &= \boldsymbol{x} \cdot \boldsymbol{x} \\ \rho &= \|\boldsymbol{x}\| \end{aligned} \tag{5}$$

we obtain, by time derivation in (5),

$$\ddot{\rho} = \frac{\dot{\boldsymbol{x}}^2 - \dot{\rho}^2 + \boldsymbol{x} \cdot \boldsymbol{F}}{\rho} \geqslant \frac{\boldsymbol{x} \cdot \boldsymbol{F}}{\rho} = F_\rho(\rho, \theta, \varphi). \tag{6}$$

Computing the term $(\boldsymbol{x} \cdot \boldsymbol{F}) \cdot \rho^{-1}$ for the $\boldsymbol{F}$ given in equation (3) we obtain

$$\begin{aligned} &\frac{\boldsymbol{x} \cdot \boldsymbol{F}}{\rho} = -G \sum_{i=1}^{N} \frac{m_i(\boldsymbol{x} - \boldsymbol{x}_i) \cdot \boldsymbol{x}}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^3 \cdot \rho} \geqslant -G \sum_{i=1}^{N} \frac{m_i(\rho^2 + \rho\rho_i)}{(\rho - \rho_i)^3 \cdot \rho} \\ &\rho > \max(\rho_1, \ldots, \rho_N). \end{aligned} \tag{7}$$

Note that the denominators of (7) never vanish outside the sphere $\rho = \max(\rho_1, \ldots, \rho_N)$. Therefore, equation (6) implies

$$\begin{aligned} &\ddot{\rho} \geqslant -\frac{\mathrm{d}W}{\mathrm{d}\rho} \\ &W = G \int \sum_{i=1}^{N} \frac{m_i(\rho + \rho_i)}{(\rho - \rho_i)^3} \,\mathrm{d}\rho. \end{aligned} \tag{8}$$

The reader can easily check that the differential equation

$$\begin{aligned} &\ddot{x} = -\frac{\mathrm{d}W}{\mathrm{d}x} \\ &W(x) = G \sum_{i=1}^{N} \int \frac{m_i(x + \rho_i)}{(x - \rho_i)^3} \\ &x > \max(\rho_1, \ldots, \rho_N) \end{aligned} \tag{9}$$

possesses unbounded solutions $\forall \boldsymbol{x}_0 > \rho_i$ when $\dot{\boldsymbol{x}}_0$ is large enough. One has just to draw $W(x)$ versus x and note that $W(x)$ is bounded when $x \to +\infty$.

The existence of unbounded solutions to equation (9) implies, via (8), that there are unbounded solutions of equation (4) when $\rho_0 > \max(\rho_1, \ldots, \rho_N)$ and $\dot{\rho}_0$ is sufficiently high. Therefore, the force field (3) admits escape solutions.

Case 2.2

Let us now assume that the unit mass M is subjected to the attraction of a continuous, but finite, mass which is distributed over a compact region $C \subset \mathbb{R}^3$. Note that M can also escape to infinity under appropriate initial conditions. The same result is obtained when M is attracted by a finite number of masses distributed over the compact regions $C_1, \ldots, C_N$. The details of the proof will not be given as they are elementary.

The gravitational potential $V(x, y, z)$ at $\boldsymbol{x} \notin C$ is given by

$$V = -G \iiint_C \frac{d(\boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|} \, \mathrm{d}V' \qquad \boldsymbol{x}' \in C \tag{10}$$

where $d(\boldsymbol{x}')$ denotes the density of the distribution of matter inside C. We assume continuity of $d(\boldsymbol{x}')$. The integral in (10) is convergent for any $\boldsymbol{x} \notin C$, as C is a compact set.

On the other hand, equation (4) can be written as

$$\ddot{\boldsymbol{x}} = -\nabla V \tag{11}$$

where V is given in (10). Note that ∇V is well defined since $\|\boldsymbol{x} - \boldsymbol{x}'\|^{-1}$ is a class-one function in $\boldsymbol{x}$ and the region C is compact. We obtain in this way the convergent expression for $\boldsymbol{F}(x, y, z)$,

$$\boldsymbol{F}(\boldsymbol{x}) = -G \iiint_C \frac{(\boldsymbol{x} - \boldsymbol{x}') \, d(x')}{\|\boldsymbol{x} - \boldsymbol{x}'\|^3} \, \mathrm{d}V' \qquad \boldsymbol{x} \notin C. \tag{12}$$

Let us now obtain a bound for the term $(\boldsymbol{F} \cdot \boldsymbol{x}) \cdot \rho^{-1}$ of equation (6):

$$\begin{aligned} (\boldsymbol{F} \cdot \boldsymbol{x}) \cdot \rho^{-1} &= -G \iiint_C \frac{(\boldsymbol{x} - \boldsymbol{x}') \cdot \boldsymbol{x} \cdot d(\boldsymbol{x}')}{\rho \|\boldsymbol{x} - \boldsymbol{x}'\|^3} \, \mathrm{d}V' \\ &\geqslant -G \iiint_C \frac{(\rho^2 + \rho\rho') \, d(\boldsymbol{x}')}{\rho(\rho - \rho')^3} \, \mathrm{d}V' = -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho} \end{aligned} \tag{13}$$

where $W(\rho)$ is defined by

$$W(\rho) = -G \int \left(\iiint_C \frac{(\rho + \rho') \, d(\boldsymbol{x}')}{(\rho - \rho')^3} \, \mathrm{d}V' \right) \mathrm{d}\rho. \tag{14}$$

Note that all the integrals in (13) and (14) are well defined for $\rho > \rho'(C)$.

We can therefore write†

$$\ddot{\rho} \geqslant \frac{\boldsymbol{x} \cdot \boldsymbol{F}}{\rho} \geqslant -\frac{\mathrm{d}W(\rho)}{\mathrm{d}\rho}. \tag{15}$$

Proceeding now as in Case 2.1, escape to infinity is ensured if we can show that $W(\rho)$ is bounded when $\rho \to +\infty$. This is easy to prove since the function $(\rho + \rho')/(\rho - \rho')^3$ behaves like ρ^{-2} for $\rho \to +\infty$ and the integration domain C in (14) is compact. Therefore, the boundedness of $W(\rho)$ is ensured, and so is escape to infinity under the action of the force field (12).

† For the mathematical justification concerning the commutation of integrals, series and the derivative operator, see [9].

3. Escape in the presence of an infinite system of discrete masses

In this section we study the possibility of escape to infinity under the attraction of the mass distribution

$$(m_i, \boldsymbol{x}_i) \qquad \sum_i m_i = \text{finite}. \tag{16}$$

First of all, additional assumptions are made in order to make V and $\boldsymbol{F}$ well defined functions for every $\boldsymbol{x} \neq \boldsymbol{x}_i$.

These assumptions are

$$\|\boldsymbol{x}_i\| \to +\infty \tag{17}$$

and

$$\begin{aligned} &\sum_1^\infty m_i \|\boldsymbol{x}_i\|^{-1} = \text{finite} \\ &\sum_1^\infty m_i \|\boldsymbol{x}_i\|^{-2} = \text{finite}. \end{aligned} \tag{18}$$

It is easy to see that (17) and (18) imply that the series defining V and $\boldsymbol{F}$, that is

$$\begin{aligned} V(\boldsymbol{x}) &= -G \sum_{i=1}^\infty \frac{m_i}{\|\boldsymbol{x} - \boldsymbol{x}_i\|} \\ \boldsymbol{F}(\boldsymbol{x}) &= -G \sum_{i=1}^\infty \frac{m_i(\boldsymbol{x} - \boldsymbol{x}_i)}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^3} \end{aligned} \tag{19}$$

are convergent (in fact, absolutely convergent), since (by (17)) its terms, when $i \to +\infty$, behave thus:

$$\begin{aligned} &\frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_i\|} \sim \frac{1}{\|\boldsymbol{x}_i\|} \\ &\frac{\|x - x_i\|}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^3} \leqslant \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^2} \sim \frac{1}{\|\boldsymbol{x}_i\|^2} \\ &\frac{\|y - y_i\|}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^3} \leqslant \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^2} \sim \frac{1}{\|\boldsymbol{x}_i\|^2} \\ &\frac{\|z - z_i\|}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^3} \leqslant \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^2} \sim \frac{1}{\|\boldsymbol{x}_i\|^2} \end{aligned} \tag{20}$$

and, by (18), this implies their convergence [9].

Note that V and $\boldsymbol{F}$ are well defined by (19) even if $\sum m_i = +\infty$.

Observe that equation (17) implies

$$\|\boldsymbol{x} - \boldsymbol{x}_i\| > a(\boldsymbol{x}) > 0 \qquad \forall i. \tag{21}$$

Equation (21) shows that the mass distribution $(m_i, \boldsymbol{x}_i)$ cannot accumulate around any $\boldsymbol{x} \neq \boldsymbol{x}_i$.

Note also that V and $\boldsymbol{F}$, as defined in (19), do satisfy the usual relation $\boldsymbol{F} = -\nabla V$ due to the uniform convergence in $\boldsymbol{x}$ of the series for $\boldsymbol{F}(\boldsymbol{x})$, which follows immediately from equation (20).

Let us now give two examples of mass distributions satisfying equations (16)–(18) for which escape to infinity is possible.

Case 3.1

Let $\boldsymbol{x}_i = (x_i, y_i, z_i)$ and $z_i < 0, \forall i$. We show that the unit mass can escape to infinity in the region $z > 0$.

In fact, we can write for the z-coordinate of the unit mass [9]

$$\ddot{z} = -G\sum_{i=1}^{\infty}\frac{m_i(z-z_i)}{[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2]^{3/2}} \geqslant -G\sum_{i=1}^{\infty}\frac{m_i}{(z-z_i)^2}$$
$$\geqslant -G\sum_{i=1}^{\infty}\frac{m_i}{z^2} = -\frac{\mathrm{d}W(z)}{\mathrm{d}z} \tag{22}$$
$$W(z) = -\frac{G\sum m_i}{z}$$

where we have taken into account that $(z - z_i) > 0$ in the region of space $z > 0$.

Now the function $V(z) = -(G\sum m_i)/z$ is bounded when $z \to +\infty$. Therefore, escape to infinity is ensured.

Note that our hypothesis $z_i < 0, \forall i$ includes the cases of mass distribution on a straight line or inside a cylinder. It does not include the case of masses outside a cylinder.

Case 3.2

We now assume that the mass distribution $(m_i, \boldsymbol{x}_i)$ is symmetrical with respect to the z-axis. That is, for every $\boldsymbol{x}_i = (x_i, y_i, z_i)$ there exists a $\boldsymbol{x}_i^* = (-x_i, -y_i, z_i)$ with $m_i^* = m_i$. We also assume

$$r_i^2 = x_i^2 + y_i^2 \geqslant R_0^2 \qquad \forall i \quad R_0 > 0. \tag{23}$$

That is, the masses m_i do not accumulate around the z-axis. Under these conditions we prove that escape to infinity along the z-axis is possible.

Note that the z-axis is an invariant line since

$$\boldsymbol{F}|_{z\text{-axis}} \| (z\text{-axis}). \tag{24}$$

Therefore, if the initial conditions of the unit mass are

$$\begin{aligned} \boldsymbol{x}_0 &= (0, 0, z_0) \\ \dot{\boldsymbol{x}}_0 &= (0, 0, \dot{z}_0) \end{aligned} \tag{25}$$

this mass will never leave the z-axis.

The differential equation for z is

$$\ddot{z} = -2G\sum_{i=1}^{\infty}\frac{m_i(z-z_i)}{(r_i^2+(z-z_i)^2)^{3/2}} \qquad r_i^2 = (x_i^2+y_i^2). \tag{26}$$

The factor of two in equation (26) appears since we have lumped together the attraction of the symmetrical masses m_i and m_i^*.

Now equation (26) can also be written in the form [9]

$$\begin{aligned} \ddot{z} &= -\frac{\mathrm{d}}{\mathrm{d}z}V(z) \\ V(z) &= -\sum_{i=1}^{\infty}\frac{2Gm_i}{[r_i^2+(z-z_i)^2]^{1/2}}. \end{aligned} \tag{27}$$

The potential $V(z)$ is bounded when $z \to +\infty$. So by equation (23) we can write

$$\sum_{i=1}^{\infty} \frac{m_i}{[r_i^2 + (z - z_i)^2]^{1/2}} \leqslant \sum_{i=1}^{\infty} \frac{m_i}{r_i} \leqslant \frac{\sum m_i}{R_0}. \tag{28}$$

Therefore, the unit mass M can escape to infinity.

4. Relativistic escape

We now prove that escape to infinity occurs for all the cases studied in sections 2 and 3.

The differential equations of the relativistic motion of a unit mass under the force field $\boldsymbol{F}(\boldsymbol{x})$ are

$$\ddot{\boldsymbol{x}} = (1 - \dot{\boldsymbol{x}}^2)^{1/2}\, M\, \boldsymbol{F}(\boldsymbol{x}) \tag{29}$$

with M being the matrix

$$M = \begin{pmatrix} 1 - \dot{x}^2 & -\dot{x}\dot{y} & -\dot{x}\dot{z} \\ -\dot{x}\dot{y} & 1 - \dot{y}^2 & -\dot{y}\dot{z} \\ -\dot{x}\dot{z} & -\dot{y}\dot{z} & 1 - \dot{z}^2 \end{pmatrix}. \tag{30}$$

Proceeding as in the previous sections we obtain

$$\ddot{\rho} \geqslant (1 - \dot{\boldsymbol{x}}^2)^{1/2}\, \boldsymbol{x}\, M \cdot \frac{\boldsymbol{F}}{\rho}. \tag{31}$$

We now examine the examples from section 2.

From equations (3) and (31) we obtain

$$\begin{aligned} \ddot{\rho} &\geqslant -(1 - \dot{\boldsymbol{x}}^2)^{1/2} \frac{G}{\rho} \sum_{i=1}^{\infty} \frac{\boldsymbol{x} \cdot M(\boldsymbol{x} - \boldsymbol{x}_i)}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^3} \\ &\geqslant -(1 - \dot{\boldsymbol{x}}^2)^{1/2} \frac{G}{\rho} \sum_{i=1}^{\infty} \frac{\rho^2 + \rho\rho_i}{(\rho - \rho_i)^3} \geqslant -G \sum_{i=1}^{\infty} \frac{\rho + \rho_i}{(\rho - \rho_i)^3}. \end{aligned} \tag{32}$$

At this point the reasoning to prove that escape to infinity is possible is the same as in section 2.

From equations (12) and (31) we obtain

$$\begin{aligned} \ddot{\rho} &\geqslant -(1 - \dot{\boldsymbol{x}}^2)^{1/2} \frac{G}{\rho} \iiint_C \frac{\boldsymbol{x} \cdot M(\boldsymbol{x} - \boldsymbol{x}')\, d(\boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|^3} \, \mathrm{d}V' \\ &\geqslant -(1 - \dot{\boldsymbol{x}}^2)^{1/2} \frac{G}{\rho} \iiint_C \frac{(\rho^2 + \rho\rho')\, d(\boldsymbol{x}')}{\|\boldsymbol{x} - \boldsymbol{x}'\|^3} \, \mathrm{d}V' \\ &\geqslant -G \iiint_C \frac{(\rho + \rho')\, d(\boldsymbol{x}')}{(\rho - \rho')^3} \, \mathrm{d}V'. \end{aligned} \tag{33}$$

At this point escape to infinity is proved as in section 2.

Let us now examine the examples from section 3. In the case of equation (22) the relativistic equation is

$$\ddot{z} = -(1 - \dot{\boldsymbol{x}}^2)^{1/2} G \sum_{i=1}^{\infty} m_i \frac{-(x - x_i)\dot{x}\dot{z} - (y - y_i)\dot{y}\dot{z} + (z - z_i)(1 - \dot{z}^2)}{[(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{3/2}}. \tag{34}$$

Now we can write

$$\begin{aligned}
\frac{(x-x_i)\dot{x}\dot{z}}{[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2]^{3/2}} &\geqslant \frac{-1}{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2} \geqslant \frac{-1}{(z-z_i)^2} \\
\frac{(y-y_i)\dot{y}\dot{z}}{[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2]^{3/2}} &\geqslant \frac{-1}{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2} \geqslant \frac{-1}{(z-z_i)^2} \\
\frac{-(z-z_i)(1-\dot{z}^2)}{[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2]^{3/2}} &\geqslant \frac{-1}{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2} \geqslant \frac{-1}{(z-z_i)^2}
\end{aligned} \tag{35}$$

and therefore, by equations (34) and (35), the following inequality can be written (remember that $z_i < 0\ \forall i$):

$$\begin{aligned}
\ddot{z} &\geqslant -G(1-\dot{\boldsymbol{x}}^2)^{1/2}\sum_{i=1}^{\infty}\frac{3m_i}{(z-z_i)^2} \geqslant -3G\sum_{i=1}^{\infty}\frac{m_i}{\boldsymbol{z}^2} = -\frac{\mathrm{d}W}{\mathrm{d}z} \\
W(z) &= \frac{-3G\sum_{i=1}^{\infty}m_i}{z} \qquad \text{bounded for} \quad z \to +\infty.
\end{aligned} \tag{36}$$

Therefore, escape to infinity is possible.

In the case of equation (26) its relativistic counterpart is

$$\ddot{z} = -2(1-\dot{z}^2)^{3/2}G\sum_{i=1}^{\infty}\frac{m_i(z-z_i)}{[r_i^2+(z-z_i)^2]^{3/2}} \tag{37}$$

and we can write

$$\ddot{z} \geqslant -2G\sum_{i=1}^{\infty}\frac{m_i(z-z_i)}{[r_i^2+(z-z_i)^2]^{3/2}} \tag{38}$$

and continue as in Case 3.2. Therefore, escape to infinity is possible.

These examples might lead one to conjecture that if the following limit exists and is not zero

$$\lim_{\rho\to+\infty}\frac{\boldsymbol{x}\cdot\boldsymbol{F}}{\boldsymbol{x}\cdot M\boldsymbol{F}} \tag{39}$$

then relativistic escape is implied by the existence of the classical, non-relativistic one.

5. Final remarks

We have seen that escape to infinity can be discovered without solving the equations of motion $\ddot{\boldsymbol{x}} = -\nabla V$ and some examples, with attracting masses of finite total mass, admitting escape to infinity have been given. To the best of the authors' knowledge there remains the open problem of finding a mass configuration $(m_i, \boldsymbol{x}_i)$ satisfying equation (18) not admitting escape; that is, all the trajectories of a unit mass under the action of the masses should be bounded. Even a two-dimensional example would be interesting.

Other open problems are these: is there escape when infinite masses lie outside a cylinder, or if the infinite distribution of masses lies inside the three cylinders

$$x^2+y^2 \leqslant 1 \qquad x^2+z^2 \leqslant 1 \qquad y^2+z^2 \leqslant 1\ ? \tag{40}$$

Note that the techniques of Case 3.1 do not apply in this last case since any plane cuts the region defined by (40).

Finally, we give an example of a mass distribution with $\sum m_i = +\infty$ for which escape is possible.

This distribution is given by

$$m_i = 1 \qquad \boldsymbol{x}_i = (0, 0, -i^2) \tag{41}$$

and it satisfies equation (18).

The z-axis is an invariant line, and the motion along it is given by [9]

$$\begin{aligned} \ddot{z} &= -G \sum_{i=1}^{\infty} \frac{1}{(z+i^2)^2} = -\frac{\mathrm{d}V(z)}{\mathrm{d}z} \\ V(z) &= -G \sum_{i=1}^{\infty} \frac{1}{z+i^2} \qquad z > 0. \end{aligned} \tag{42}$$

Note that $V(z)$ is bounded since

$$\sum_{i=1}^{\infty} \frac{1}{z+i^2} \leqslant \sum_{i=1}^{\infty} \frac{1}{i^2} = \text{finite}. \tag{43}$$

Therefore, escape to infinity along the z-axis ($z > 0$) is ensured.

The case of an infinite number of attracting *compact bodies* has not been considered since its treatment is similar to the case (considered in section 3) of a system of pointlike attracting centres.

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